



Enumerating 1324-avoiders with few inversions

Emil Verkama – KTH Royal Institute of Technology

NORCOM 2025, Einar Fest

Based on joint work with

Svante Linusson – KTH Royal Institute of Technology

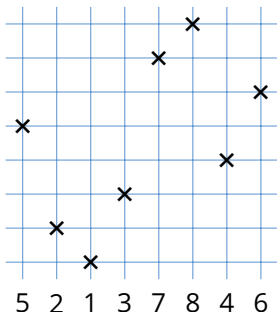
Enumerating 1324-avoiders with few inversions

arXiv: 2408.15075



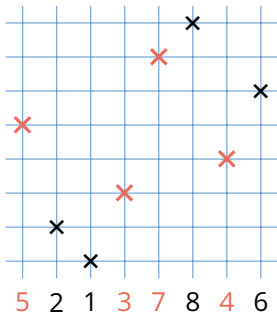
Pattern avoiding permutations

A permutation $\pi \in S_n$ contains a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p . Otherwise π avoids p .



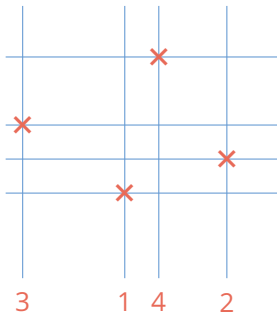
Pattern avoiding permutations

A permutation $\pi \in S_n$ contains a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p . Otherwise π avoids p .



Pattern avoiding permutations

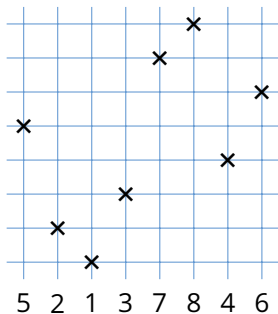
A permutation $\pi \in S_n$ contains a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p . Otherwise π avoids p .



π contains 3142

Pattern avoiding permutations

A permutation $\pi \in S_n$ contains a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p . Otherwise π avoids p .

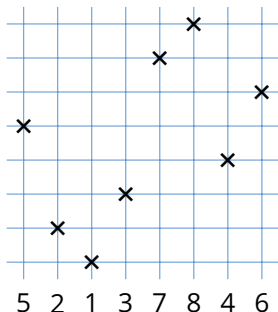


π contains 3142

π avoids 1324

Pattern avoiding permutations

A permutation $\pi \in S_n$ contains a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p . Otherwise π avoids p .



π contains 3142

π avoids 1324

We write $\text{Av}(p)$ for the set of all permutations that avoid p ,

$$\text{Av}_n(p) = \text{Av}(p) \cap S_n \quad \text{and} \quad \text{av}_n(p) = |\text{Av}_n(p)|.$$



History

Early history

MacMahon (1915) showed that

$$\text{av}_n(123) = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Early history

MacMahon (1915) showed that

$$\text{av}_n(123) = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Erdős-Szekeres theorem (1935): any permutation of length at least $(a-1)(b-1)+1$ contains the pattern

$$[1, 2, \dots, a] \quad \text{or} \quad [b, b-1, \dots, 1].$$

Early history

MacMahon (1915) showed that

$$\text{av}_n(123) = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Erdős–Szekeres theorem (1935): any permutation of length at least $(a-1)(b-1)+1$ contains the pattern

$$[1, 2, \dots, a] \quad \text{or} \quad [b, b-1, \dots, 1].$$

Knuth (1968) showed that the *stack-sortable* permutations are exactly the 231-avoiders, and $\text{av}_n(231) = C_n$.

The Renaissance

Major actors:

The Renaissance

Major actors:

- **The Steingrímsson group.**

The Renaissance

Major actors:

- **The Steingrímsson group.**
- (Many others.)

The Renaissance

Major actors:

- **The Steingrímsson group.**
- (Many others.)

Developments:

- The Marcus–Tardos theorem (2004).
- Enumeration of $\text{Av}(1234)$ and $\text{Av}(1342)$.
- Generalizations: vincular patterns, mesh patterns, ...
- Books: Bóna's *Combinatorics of Permutations* (2004), Kitaev's *Patterns in Permutations and Words* (2011).
- Annual conference *Permutation Patterns* (2003–present).



A Big Problem

The questions we ask

Holy Grail. Enumerate $\text{av}_n(p)$ for some pattern p .

The questions we ask

Holy Grail. Enumerate $\text{av}_n(p)$ for some pattern p .

Gessel (1990):

$$\text{av}_n(1234) = 2 \sum_{k=0}^n \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)}.$$

The questions we ask

Holy Grail. Enumerate $av_n(p)$ for some pattern p .

Gessel (1990):

$$av_n(1234) = 2 \sum_{k=0}^n \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)}.$$

Bóna (1997):

$$\begin{aligned} av_n(1342) = & (-1)^{n-1} \cdot \frac{7n^2 - 3n - 2}{2} \\ & + 3 \sum_{k=2}^n (-1)^{n-k} \cdot 2^{k+1} \cdot \frac{(2k-4)!}{k!(k-2)!} \cdot \binom{n-i+2}{2}. \end{aligned}$$

The questions we ask

Big Problem. Enumerate $av_n(1324)$.

This question is still wide open. Can we ask something simpler?

The questions we ask

Big Problem. Enumerate $\text{av}_n(1324)$.

This question is still wide open. Can we ask something simpler?

Theorem (Marcus–Tardos, 2004). For any pattern p , the *Stanley–Wilf limit*

$$L(p) := \lim_{n \rightarrow \infty} \text{av}_n(p)^{1/n}$$

exists and is finite.

Examples: $L(132) = 4$, $L(1234) = 9$, $L(1342) = 8$.

What is known

Known bounds for $L(1324)$.

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012)		16
Bóna (2014)		13.93
Bóna (2015)		13.74
Bevan (2015)	9.81	
Bevan–Brignall–Elvey Price–Pantone (2020)	10.27	13.5

What is known

Known bounds for $L(1324)$.

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012)		16
Bóna (2014)		13.93
Bóna (2015)		13.74
Bevan (2015)	9.81	
Bevan–Brignall–Elvey Price–Pantone (2020)	10.27	13.5

An innocent conjecture

Inversion = 21-pattern.

$$\text{Av}_n^k(p) = \{\pi \in \text{Av}_n(p) : \text{inv}(\pi) = k\}, \quad \text{av}_n^k(p) = |\text{Av}_n^k(p)|.$$

An innocent conjecture

Inversion = 21-pattern.

$$\text{Av}_n^k(p) = \{\pi \in \text{Av}_n(p) : \text{inv}(\pi) = k\}, \quad \text{av}_n^k(p) = |\text{Av}_n^k(p)|.$$

Conjecture (Claesson–Jelínek–Steingrímsson, 2012).

$$\text{av}_n^k(1324) \leq \text{av}_{n+1}^k(1324)$$

for all k, n , i.e. 1324 is *inversion monotone*.

Corollary. $L(1324) < 13.002$.

Another view

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

Numbers $av_n^k(1324)$

CJS conjecture: increasing

Another view

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

Numbers $av_n^k(1324)$

CJS conjecture: increasing

Claesson-Jelínek-Steingrímsson (2012) finds the blue sequence.

Another view

Numbers $av_n^k(1324)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

CJS conjecture: increasing

Claesson-Jelínek-Steingrímsson (2012) finds the blue sequence.

Linusson-V. (2025) proves the conjecture in the red region.

Another view

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

↙ CJS conjecture: nonnegative

Another view

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

qjs conjecture: nonnegative

Another view

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

CJS conjecture: nonnegative

A constant sequence appears in the red region.

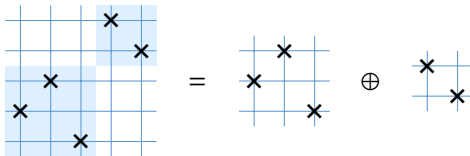
Why?



Decomposability

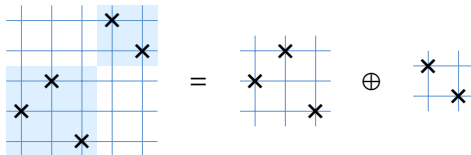
Decomposable permutations

Permutations can be decomposed with respect to the *direct sum*:



Decomposable permutations

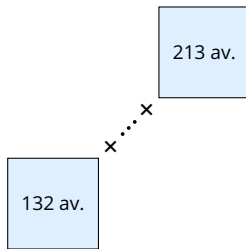
Permutations can be decomposed with respect to the *direct sum*:



Fact. A decomposable $\pi \in S_n$ avoids 1324 if and only if

$$\pi = \sigma \oplus \text{id} \oplus \tau,$$

where $\sigma \in \text{Av}(132)$, $\tau \in \text{Av}(213)$.

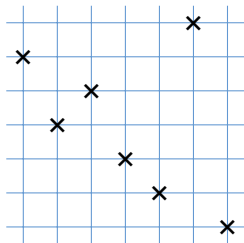


The limit sequence

Fact. If $n \geq k + 1$, then

$$\text{av}_n^k(132) = p(k),$$

the number of integer partitions of k .

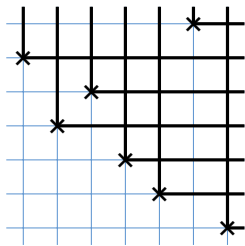


The limit sequence

Fact. If $n \geq k + 1$, then

$$\text{av}_n^k(132) = p(k),$$

the number of integer partitions of k .

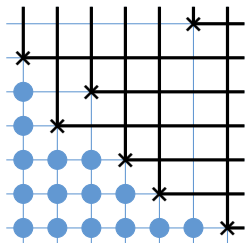


The limit sequence

Fact. If $n \geq k + 1$, then

$$\text{av}_n^k(132) = p(k),$$

the number of integer partitions of k .



The limit sequence

Fact. If $n \geq k + 1$, then

$$\text{av}_n^k(132) = p(k),$$

the number of integer partitions of k .



The limit sequence

Fact. If $n \geq k + 1$, then

$$\text{av}_n^k(132) = p(k),$$

the number of integer partitions of k .



Combining the facts: if $n \geq k + 2$, then

$$\text{av}_n^k(1324) = \sum_{i=0}^k p(i)p(k-i) = [x^k](P(x)^2).$$

This is the blue sequence

1, 2, 5, 10, 20, 36, 65, 110, 185, 300, 481, 752, 1165, ...

from the table!

Almost decomposable permutations

Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

Almost decomposable permutations

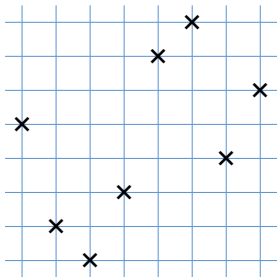
Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

Definition (Linusson-V. 2025).

$\pi \in S_n$ is *almost decomposable* if it is indecomposable, but

$$\text{comp}(\pi \setminus e) \geq 2.$$

for at least one $e \in \{1, n, \pi_1, \pi_n\}$.



Almost decomposable permutations

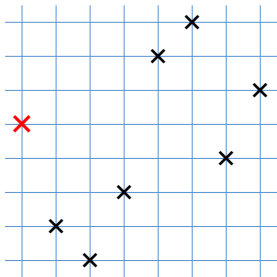
Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

Definition (Linusson-V. 2025).

$\pi \in S_n$ is *almost decomposable* if it is indecomposable, but

$$\text{comp}(\pi \setminus e) \geq 2.$$

for at least one $e \in \{1, n, \pi_1, \pi_n\}$.



Almost decomposable permutations

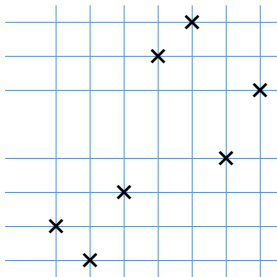
Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

Definition (Linusson-V. 2025).

$\pi \in S_n$ is *almost decomposable* if it is indecomposable, but

$$\text{comp}(\pi \setminus e) \geq 2.$$

for at least one $e \in \{1, n, \pi_1, \pi_n\}$.



Almost decomposable permutations

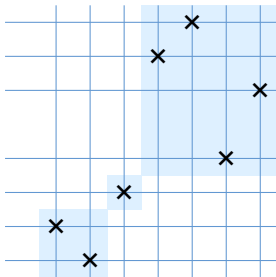
Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

Definition (Linusson-V. 2025).

$\pi \in S_n$ is *almost decomposable* if it is indecomposable, but

$$\text{comp}(\pi \setminus e) \geq 2.$$

for at least one $e \in \{1, n, \pi_1, \pi_n\}$.



The key idea

Theorem (Linusson–V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

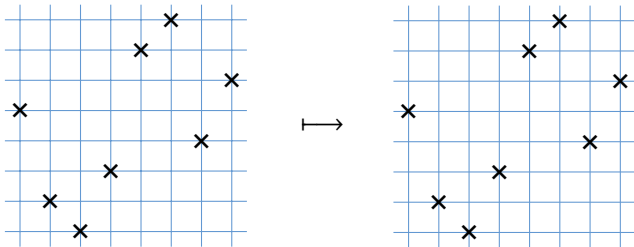
This is exactly the red region in the table!

The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:

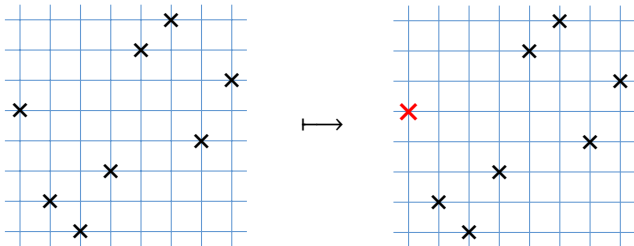


The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:

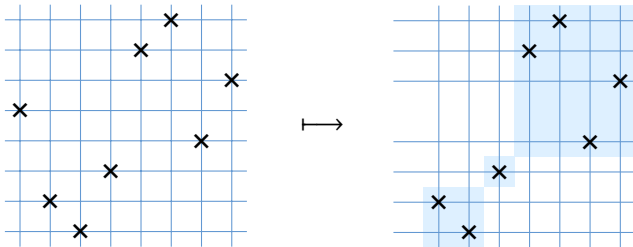


The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:

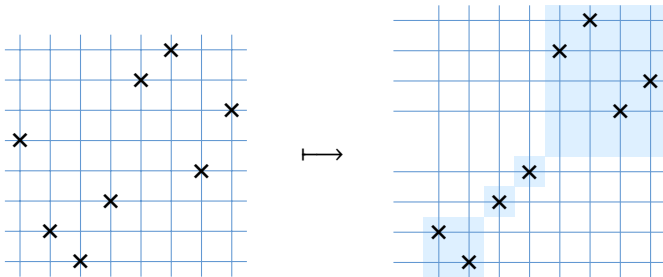


The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:

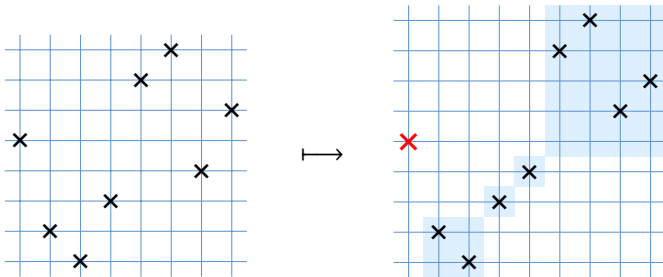


The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:

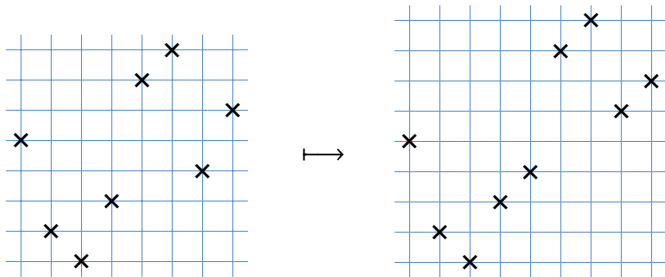


The key idea

Theorem (Linusson-V. 2025). If $n \geq \frac{k+7}{2}$, then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable.

This is exactly the **red region** in the table!

We can now build an injection $\text{Av}_n^k(1324) \rightarrow \text{Av}_{n+1}^k(1324)$:



The harvest

Theorem (Linusson-V. 2025). For all $n \geq \frac{k+7}{2}$,

$$\text{av}_n^k(1324) = a(k) - 4a(k - n + 1) - 6 \sum_{i=0}^{k-n} a(i),$$

where $a(k) = \sum_{i=0}^k p(i)p(k-i)$.

The harvest

Theorem (Linusson-V. 2025). For all $n \geq \frac{k+7}{2}$,

$$\text{av}_n^k(1324) = a(k) - 4a(k - n + 1) - 6 \sum_{i=0}^{k-n} a(i),$$

where $a(k) = \sum_{i=0}^k p(i)p(k-i)$.

Corollary. The conjecture holds when $n \geq \frac{k+7}{2}$.

Open problems.

- Prove the full conjecture. Can our method be extended?

Open problems.

- Prove the full conjecture. Can our method be extended?
- Does almost-decomposability have other applications?

Open problems.

- Prove the full conjecture. Can our method be extended?
- Does almost-decomposability have other applications?
- Other distributional properties of inv over $\text{Av}(1324)$?

For example, is

$$\text{av}_n^0(1324), \text{av}_n^1(1324), \dots, \text{av}_n^{\binom{n}{2}}(1324)$$

unimodal?

Thank you, and Happy Birthday to Einar!

Open problems.

- Prove the full conjecture. Can our method be extended?
- Does almost-decomposability have other applications?
- Other distributional properties of inv over $\text{Av}(1324)$?

For example, is

$$\text{av}_n^0(1324), \text{av}_n^1(1324), \dots, \text{av}_n^{\binom{n}{2}}(1324)$$

unimodal?

Based on joint work with

Svante Linusson – KTH Royal Institute of Technology

Enumerating 1324-avoiders with few inversions

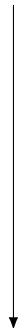
arXiv: 2408.15075



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

Numbers $av_n^k(1324)$

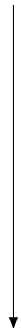
qjs conjecture: increasing



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

Numbers $av_n^k(1324)$

qjs conjecture: increasing



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
3	1	2	2	1																
4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402

Numbers $av_n^k(1324)$

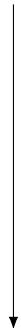
qjs conjecture: increasing



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

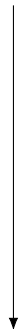
CJS conjecture: nonnegative



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

CJS conjecture: nonnegative



$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
2	0	1	2	1																
3	0	0	3	5	5	3	1													
4	0	0	0	4	11	17	19	15	9	4	1									
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1				
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

CJS conjecture: nonnegative