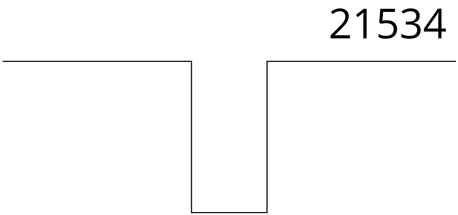
Inversions in the 1324-avoiders

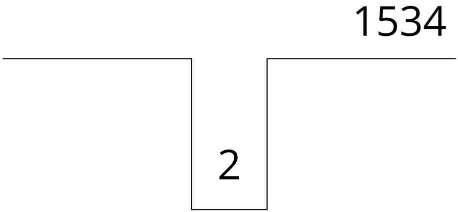
Emil Verkama – KTH Royal Institute of Technology

Licentiate seminar, 28 April 2026

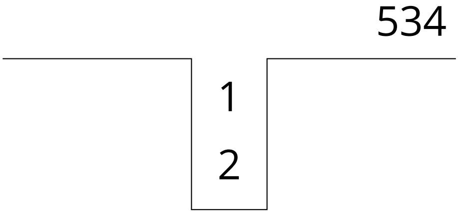
Which permutations can be sorted by a single stack?



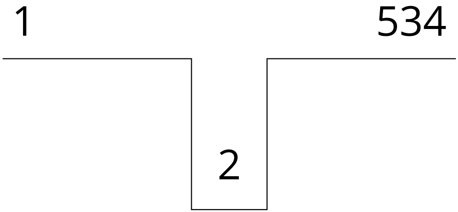
Which permutations can be sorted by a single stack?



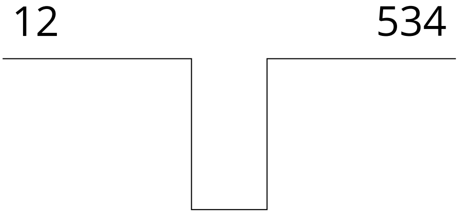
Which permutations can be sorted by a single stack?



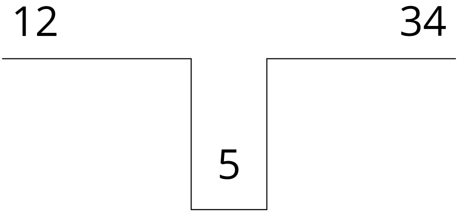
Which permutations can be sorted by a single stack?



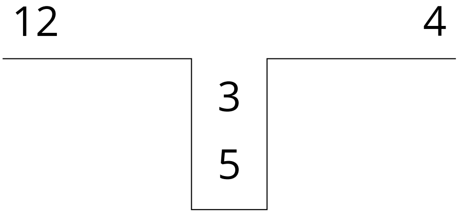
Which permutations can be sorted by a single stack?



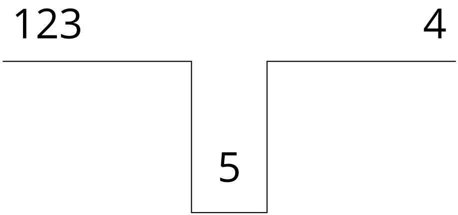
Which permutations can be sorted by a single stack?



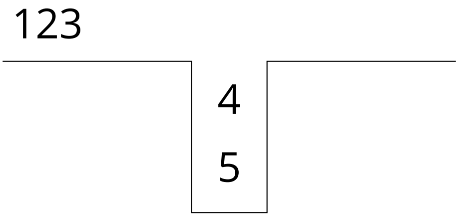
Which permutations can be sorted by a single stack?



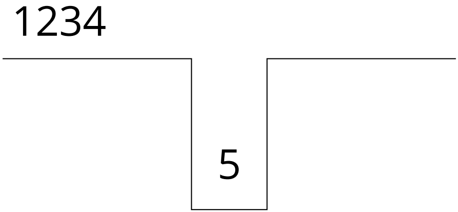
Which permutations can be sorted by a single stack?



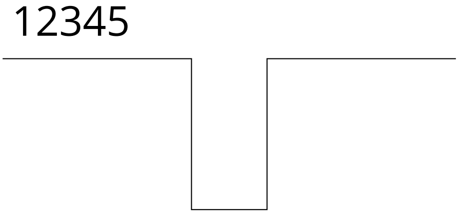
Which permutations can be sorted by a single stack?



Which permutations can be sorted by a single stack?

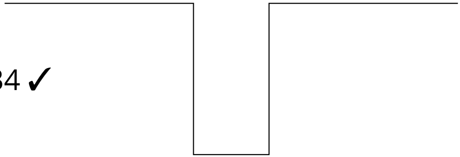


Which permutations can be sorted by a single stack?



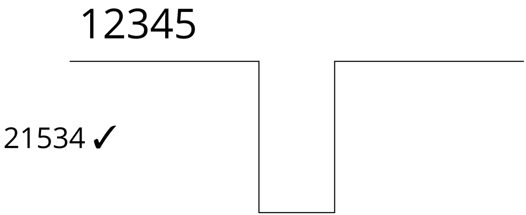
Which permutations can be sorted by a single stack?

12345



21534 ✓

Which permutations can be sorted by a single stack?

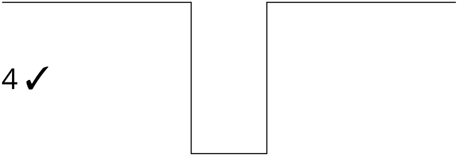


Fact (Knuth, 1968)**.** A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

Which permutations can be sorted by a single stack?

12345

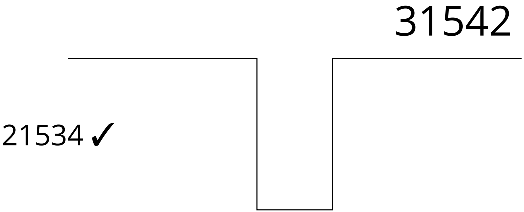
21534 ✓



Fact (Knuth, 1968)**.** A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

Which permutations can be sorted by a single stack?

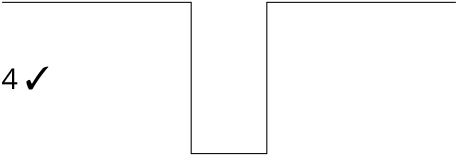


Fact (Knuth, 1968)**.** A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

Which permutations can be sorted by a single stack?

31542

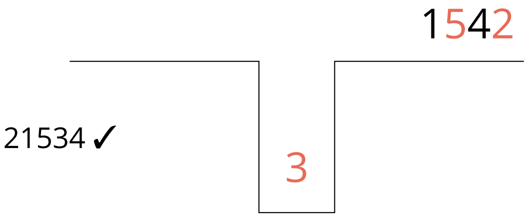


21534 ✓

Fact (Knuth, 1968). A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

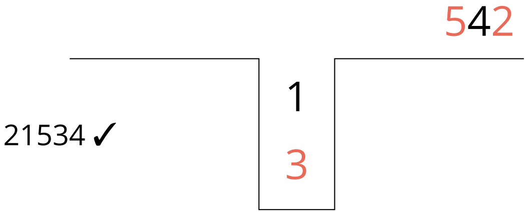
Which permutations can be sorted by a single stack?



Fact (Knuth, 1968). A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

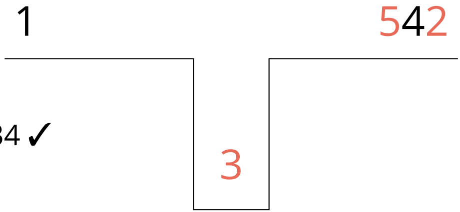
Which permutations can be sorted by a single stack?



Fact (Knuth, 1968). A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

Which permutations can be sorted by a single stack?

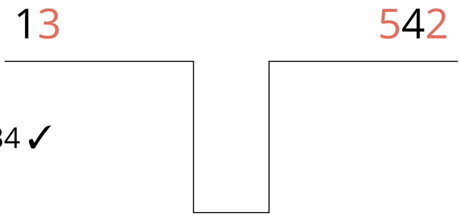


21534 ✓

Fact (Knuth, 1968). A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

Which permutations can be sorted by a single stack?

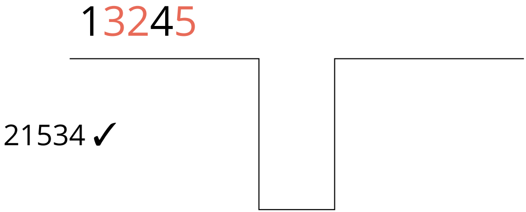


21534 ✓

Fact (Knuth, 1968). A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

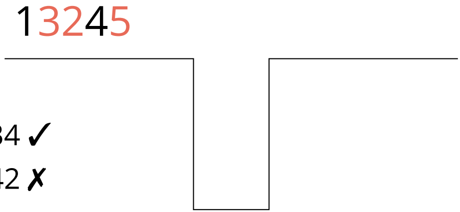
Which permutations can be sorted by a single stack?



Fact (Knuth, 1968)**.** A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

Which permutations can be sorted by a single stack?



21534 ✓

31542 ✗

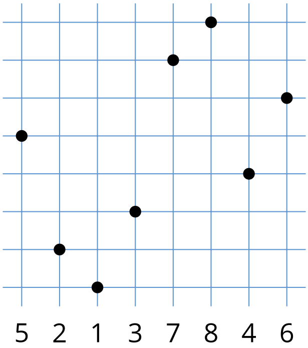
Fact (Knuth, 1968)**.** A permutation π is stack-sortable if and only if it has no subsequence a, b, c such that $c < a < b$.

231-pattern

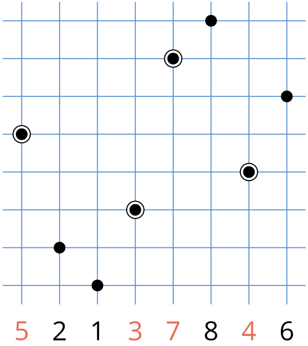
Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ.

Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ. Otherwise, π **avoids** σ.

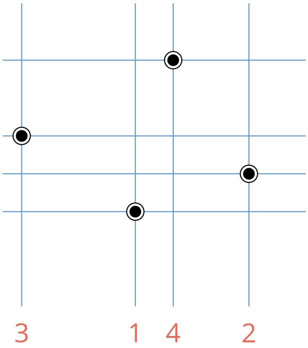
Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ. Otherwise, π **avoids** σ.



Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ. Otherwise, π **avoids** σ.

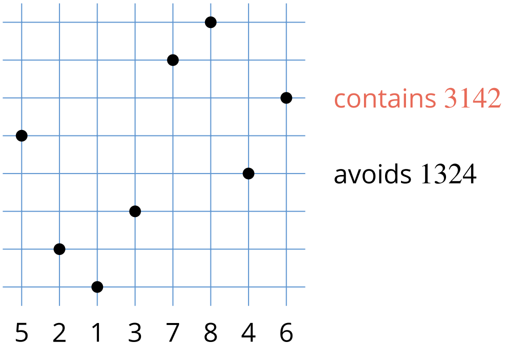


Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ. Otherwise, π **avoids** σ.



contains 3142

Definition. A permutation π **contains** a pattern σ if π has a subsequence whose elements are in the same relative order as σ. Otherwise, π **avoids** σ.



Definition. $\mathrm{Av}(\sigma)$ = set of all σ-avoiding permutations, $\mathrm{Av}_n(\sigma) = \mathrm{Av}(\sigma) \cap S_n$, and $\mathrm{av}_n(\sigma) = |\mathrm{Av}_n(\sigma)|$.

Example. Stack-sortable permutations = $\mathrm{Av}(231)$.

Definition. $\mathrm{Av}(\sigma)$ = set of all σ-avoiding permutations, $\mathrm{Av}_n(\sigma) = \mathrm{Av}(\sigma) \cap S_n$, and $\mathrm{av}_n(\sigma) = |\mathrm{Av}_n(\sigma)|$.

Example. Stack-sortable permutations = $\mathrm{Av}(231)$.

Definition. A *permutation class* is a set C of permutations such that if $\pi \in C$ and σ is contained in π, then $\sigma \in C$.

Very natural objects!

Definition. $\mathrm{Av}(\sigma)$ = set of all σ-avoiding permutations, $\mathrm{Av}_n(\sigma) = \mathrm{Av}(\sigma) \cap S_n$, and $\mathrm{av}_n(\sigma) = |\mathrm{Av}_n(\sigma)|$.

Example. Stack-sortable permutations = $\mathrm{Av}(231)$.

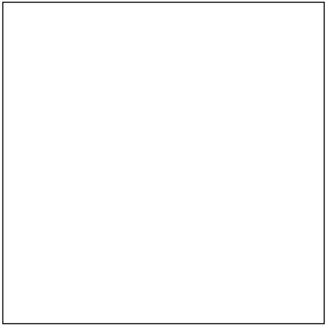
Definition. A *permutation class* is a set C of permutations such that if $\pi \in C$ and σ is contained in π, then $\sigma \in C$.

Very natural objects!

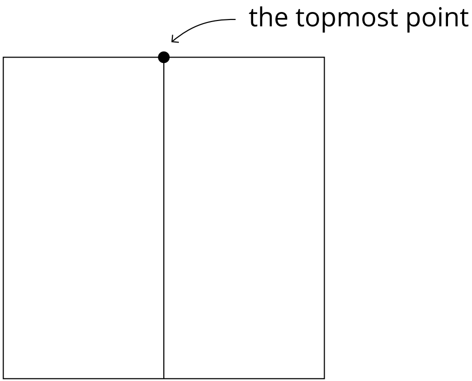
Fact. Permutation class = pattern avoidance.

The principal classes

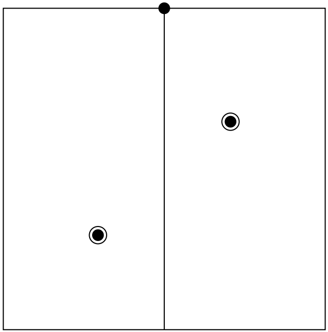
Example. The structure of a 132-avoiding permutation:



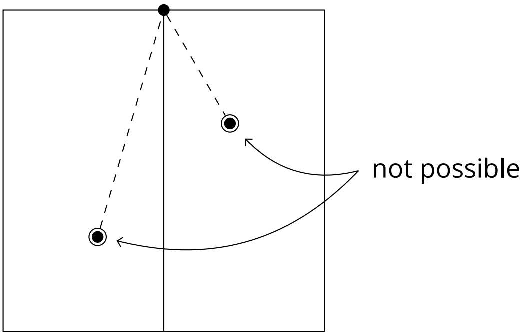
Example. The structure of a 132-avoiding permutation:



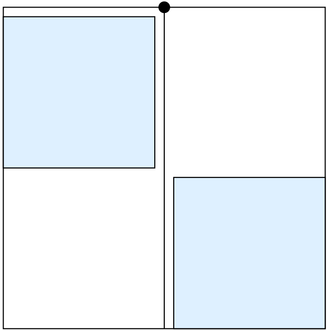
Example. The structure of a 132-avoiding permutation:



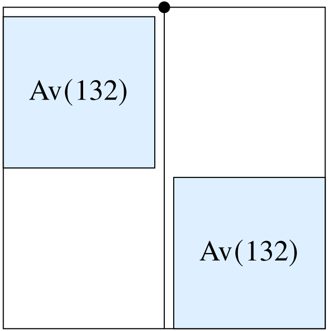
Example. The structure of a 132-avoiding permutation:



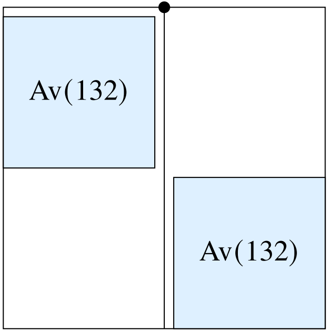
Example. The structure of a 132-avoiding permutation:



Example. The structure of a 132-avoiding permutation:



Example. The structure of a 132-avoiding permutation:



$$\mathrm{av}_n(132) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Main question. Find a formula for $\mathrm{av}_n(\sigma)$.

Main question. Find a formula for $\mathrm{av}_n(\sigma)$.

MacMahon (1915): $\mathrm{av}_n(123) = C_n$.

Erdős–Szekeres (1935): for $n \geqslant k\ell + 1$,

$$\mathrm{av}_n(12\ldots k,\ \ell\ldots 21) = 0.$$

Knuth (1968): $\mathrm{Av}(231)$ = stack-sortable, $\mathrm{av}_n(231) = C_n$.

Main question. Find a formula for $\mathrm{av}_n(\sigma)$.

MacMahon (1915): $\mathrm{av}_n(123) = C_n$.

Erdős–Szekeres (1935): for $n \geqslant k\ell + 1$,

$$\mathrm{av}_n(12\ldots k,\ \ell\ldots 21) = 0.$$

Knuth (1968): $\mathrm{Av}(231)$ = stack-sortable, $\mathrm{av}_n(231) = C_n$.

Proposition. $\mathrm{av}_n(\sigma)$ is the same for all $\sigma \in S_3$.

Sad fact. For $\sigma \in S_4$, there are three distinct cases.

Sad fact. For $\sigma \in S_4$, there are three distinct cases.

Gessel (1990):

$$\mathrm{av}_n(1234) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}.$$

Sad fact. For $\sigma \in S_4$, there are three distinct cases.

Gessel (1990):

$$\mathrm{av}_n(1234) = 2 \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)}.$$

Bóna (1997):

$$\begin{aligned}\mathrm{av}_n(1342) = {} & (-1)^{n-1} \cdot \frac{7n^2 - 3n - 2}{2} \\ & + 3 \sum_{k=2}^{n} (-1)^{n-k} \cdot 2^{k+1} \cdot \frac{(2k-4)!}{k!(k-2)!} \cdot \binom{n-k+2}{2}.\end{aligned}$$

The mystery of 1324

Embarrassing problem. What is $\mathrm{av}_n(1324)$?

The mystery of 1324

Embarrassing problem. What is $\mathrm{av}_n(1324)$?

Nobody knows. Can we ask something simpler?

The mystery of 1324

Embarrassing problem. What is $\mathrm{av}_n(1324)$?

Nobody knows. Can we ask something simpler?

Theorem (Marcus–Tardos, 2004)**.** For any pattern σ, the *Stanley–Wilf limit*

$$L(\sigma) := \lim_{n\to\infty} \mathrm{av}_n(\sigma)^{1/n}$$

exists and is finite.

Examples. $L(132) = 4$, $L(1234) = 9$, $L(1342) = 8$.

Known bounds for $L(1324)$.

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012) (**CJS**)		16
Bóna (2014)		13.93
Bóna (2015)		13.74
Bevan (2015)	9.81	
Bevan–Brignall–Elvey Price–Pantone (2020)	10.27	13.5

Known bounds for $L(1324)$.

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012) (**CJS**)		16
Bóna (2014)		13.93
Bóna (2015)		13.74
Bevan (2015)	9.81	
Bevan–Brignall–Elvey Price–Pantone (2020)	10.27	13.5

Inversions

Inversions = measure of disorder in a permutation.

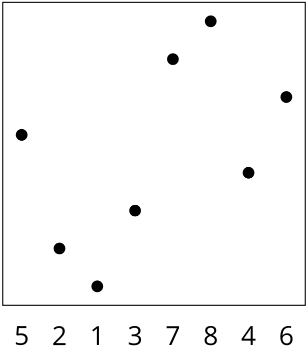
Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

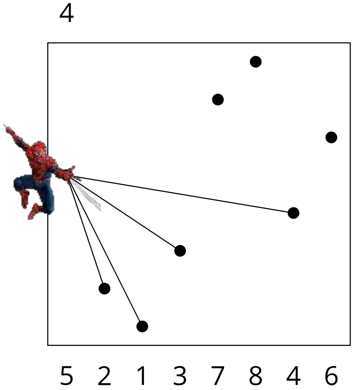
Example. $\mathrm{inv}(\pi) = 9$.



Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

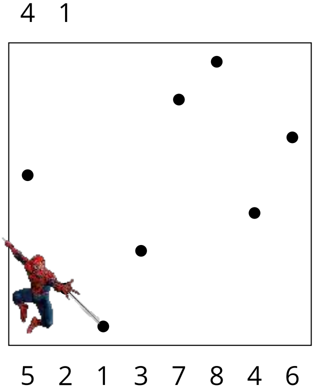
Example. $\mathrm{inv}(\pi) = 9$.



Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

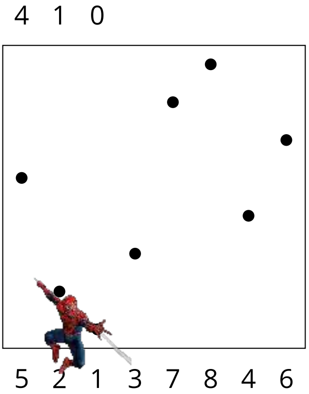
Example. $\mathrm{inv}(\pi) = 9$.



Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

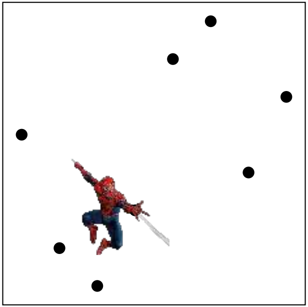


Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

4 1 0 0



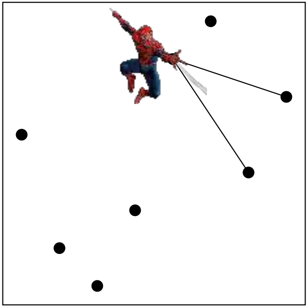
5 2 1 3 7 8 4 6

Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

4 1 0 0 2

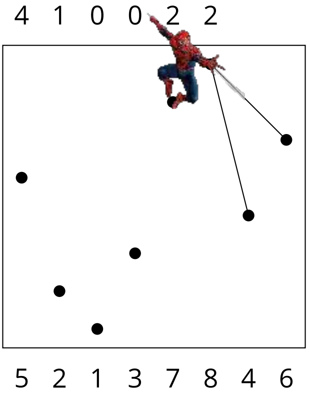


5 2 1 3 7 8 4 6

Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

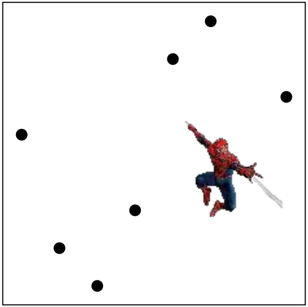


Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

4 1 0 0 2 2 0



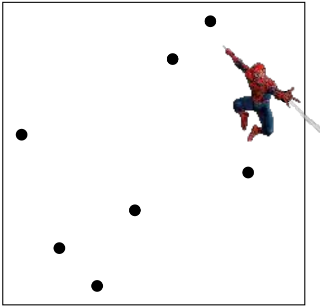
5 2 1 3 7 8 4 6

Inversions = measure of disorder in a permutation.

Inversion = 21-pattern, $\mathrm{inv}(\pi)$ = number of inversions.

Example. $\mathrm{inv}(\pi) = 9$.

4 1 0 0 2 2 0 0



5 2 1 3 7 8 4 6

Question. How do inversions behave on pattern-avoiding permutations?

Very difficult problem!

Question. How do inversions behave on pattern-avoiding permutations?

Very difficult problem!

Let $\mathrm{av}_n^k(\sigma) = |\{\pi \in \mathrm{Av}_n(\sigma) : \mathrm{inv}(\pi) = k\}|$.

We are interested in the *shape* of the distribution. Is it …

Question. How do inversions behave on pattern-avoiding permutations?

Very difficult problem!

Let $\mathrm{av}_n^k(\sigma) = |\{\pi \in \mathrm{Av}_n(\sigma) : \mathrm{inv}(\pi) = k\}|$.

We are interested in the *shape* of the distribution. Is it ...

- *unimodal* in k?

$$\mathrm{av}_n^0(\sigma) \leqslant \ldots \leqslant \mathrm{av}_n^i(\sigma) \geqslant \ldots \geqslant \mathrm{av}_n^{\binom{n}{2}}(\sigma).$$

Question. How do inversions behave on pattern-avoiding permutations?

Very difficult problem!

Let $\mathrm{av}_n^k(\sigma) = |\{\pi \in \mathrm{Av}_n(\sigma) : \mathrm{inv}(\pi) = k\}|$.

We are interested in the *shape* of the distribution. Is it …

- *unimodal* in k?

$$\mathrm{av}_n^0(\sigma) \leqslant \ldots \leqslant \mathrm{av}_n^i(\sigma) \geqslant \ldots \geqslant \mathrm{av}_n^{\binom{n}{2}}(\sigma).$$

- *monotone* in n?

$$\mathrm{av}_0^k(\sigma) \leqslant \mathrm{av}_1^k(\sigma) \leqslant \mathrm{av}_2^k(\sigma) \leqslant \ldots$$

Question. How do inversions behave on pattern-avoiding permutations?

Very difficult problem!

Let $\mathrm{av}_n^k(\sigma) = |\{\pi \in \mathrm{Av}_n(\sigma) : \mathrm{inv}(\pi) = k\}|$.

We are interested in the *shape* of the distribution. Is it ...

- *unimodal* in k?

$$\mathrm{av}_n^0(\sigma) \leqslant \ldots \leqslant \mathrm{av}_n^i(\sigma) \geqslant \ldots \geqslant \mathrm{av}_n^{\binom{n}{2}}(\sigma).$$

- *monotone* in n?

$$\mathrm{av}_0^k(\sigma) \leqslant \mathrm{av}_1^k(\sigma) \leqslant \mathrm{av}_2^k(\sigma) \leqslant \ldots$$

Example. The numbers $\mathrm{av}_n^k(132)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	1	2	1												
4	1	1	2	3	3	3	1									
5	1	1	2	3	5	5	7	7	6	4	1					
6	1	1	2	3	5	7	9	11	14	16	16	17	14	10	5	1
7	1	1	2	3	5	7	11	13	18	22	28	32	37	40	44	43
8	1	1	2	3	5	7	11	15	20	26	34	42	53	63	73	85
9	1	1	2	3	5	7	11	15	22	28	38	48	63	77	97	116
10	1	1	2	3	5	7	11	15	22	30	40	52	69	87	111	138
11	1	1	2	3	5	7	11	15	22	30	42	54	73	93	121	152
12	1	1	2	3	5	7	11	15	22	30	42	56	75	97	127	162
13	1	1	2	3	5	7	11	15	22	30	42	56	77	99	131	168
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	172
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174

Example. The numbers $\mathrm{av}_n^k(132)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	1	2	1												
4	1	1	2	3	3	3	1									
5	1	1	2	3	5	5	7	7	6	4	1					
6	1	1	2	3	5	7	9	11	14	16	16	17	14	10	5	1
7	1	1	2	3	5	7	11	13	18	22	28	32	37	40	44	43
8	1	1	2	3	5	7	11	15	20	26	34	42	53	63	73	85
9	1	1	2	3	5	7	11	15	22	28	38	48	63	77	97	116
10	1	1	2	3	5	7	11	15	22	30	40	52	69	87	111	138
11	1	1	2	3	5	7	11	15	22	30	42	54	73	93	121	152
12	1	1	2	3	5	7	11	15	22	30	42	56	75	97	127	162
13	1	1	2	3	5	7	11	15	22	30	42	56	77	99	131	168
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	172
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174

Unimodality conjectured by Stanton (1990).

Example. The numbers $\mathrm{av}_n^k(132)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	1	2	1												
4	1	1	2	3	3	3	1									
5	1	1	2	3	5	5	7	7	6	4	1					
6	1	1	2	3	5	7	9	11	14	16	16	17	14	10	5	1
7	1	1	2	3	5	7	11	13	18	22	28	32	37	40	44	43
8	1	1	2	3	5	7	11	15	20	26	34	42	53	63	73	85
9	1	1	2	3	5	7	11	15	22	28	38	48	63	77	97	116
10	1	1	2	3	5	7	11	15	22	30	40	52	69	87	111	138
11	1	1	2	3	5	7	11	15	22	30	42	54	73	93	121	152
12	1	1	2	3	5	7	11	15	22	30	42	56	75	97	127	162
13	1	1	2	3	5	7	11	15	22	30	42	56	77	99	131	168
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	172
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174

Unimodality conjectured by Stanton (1990).

Inversion monotonicity: columns are nondecreasing.

Example. The numbers $\mathrm{av}_n^k(132)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	1	2	1												
4	1	1	2	3	3	3	1									
5	1	1	2	3	5	5	7	7	6	4	1					
6	1	1	2	3	5	7	9	11	14	16	16	17	14	10	5	1
7	1	1	2	3	5	7	11	13	18	22	28	32	37	40	44	43
8	1	1	2	3	5	7	11	15	20	26	34	42	53	63	73	85
9	1	1	2	3	5	7	11	15	22	28	38	48	63	77	97	116
10	1	1	2	3	5	7	11	15	22	30	40	52	69	87	111	138
11	1	1	2	3	5	7	11	15	22	30	42	54	73	93	121	152
12	1	1	2	3	5	7	11	15	22	30	42	56	75	97	127	162
13	1	1	2	3	5	7	11	15	22	30	42	56	77	99	131	168
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	172
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174

Unimodality conjectured by Stanton (1990).

Inversion monotonicity: columns are nondecreasing.

Inversion montonicity

We say that σ is *inversion monotone* if

$$\mathrm{av}_n^k(\sigma) \leqslant \mathrm{av}_{n+1}^k(\sigma) \quad \text{for all } k, n.$$

We say that σ is *inversion monotone* if

$$\mathrm{av}_n^k(\sigma) \leqslant \mathrm{av}_{n+1}^k(\sigma) \quad \text{for all } k, n.$$

Conjecture (CJS)**.** The pattern 1324 is inversion monotone.

We say that σ is *inversion monotone* if

$$\mathrm{av}_n^k(\sigma) \leqslant \mathrm{av}_{n+1}^k(\sigma) \quad \text{for all } k, n.$$

Conjecture (CJS)**.** The pattern 1324 is inversion monotone.

Corollary. $L(1324) < 13.002.$

Recall: current best is $L(1324) \leqslant 13.5.$

We say that σ is *inversion monotone* if

$$\mathrm{av}_n^k(\sigma) \leqslant \mathrm{av}_{n+1}^k(\sigma) \quad \text{for all } k, n.$$

Conjecture (CJS)**.** The pattern 1324 is inversion monotone.

Corollary. $L(1324) < 13.002$.

Recall: current best is $L(1324) \leqslant 13.5$.

This notorious conjecture is the focus of the thesis.

Fact. 132 is inversion monotone: take $\pi \in \mathrm{Av}_n^k(132)$, set $n+1$ as the last element. This is an injection

$$\mathrm{Av}_n^k(132) \longrightarrow \mathrm{Av}_{n+1}^k(132).$$

Fact. 132 is inversion monotone: take $\pi \in \mathrm{Av}_n^k(132)$, set $n+1$ as the last element. This is an injection

$$\mathrm{Av}_n^k(132) \longrightarrow \mathrm{Av}_{n+1}^k(132).$$

In fact. If $\sigma \in S_m$ is a pattern such that $\sigma(1) \neq 1$ or $\sigma(m) \neq m$, then σ is inversion monotone.

Fact. 132 is inversion monotone: take $\pi \in \mathrm{Av}_n^k(132)$, set $n+1$ as the last element. This is an injection

$$\mathrm{Av}_n^k(132) \longrightarrow \mathrm{Av}_{n+1}^k(132).$$

In fact. If $\sigma \in S_m$ is a pattern such that $\sigma(1) \neq 1$ or $\sigma(m) \neq m$, then σ is inversion monotone.

Another fact. The increasing patterns $12\ldots\ell$ are **not** inversion monotone.

Fact. 132 is inversion monotone: take $\pi \in \mathrm{Av}_n^k(132)$, set $n+1$ as the last element. This is an injection

$$\mathrm{Av}_n^k(132) \longrightarrow \mathrm{Av}_{n+1}^k(132).$$

In fact. If $\sigma \in S_m$ is a pattern such that $\sigma(1) \neq 1$ or $\sigma(m) \neq m$, then σ is inversion monotone.

Another fact. The increasing patterns $12\ldots\ell$ are **not** inversion monotone.

So, 1324 is the first nontrivial case.

Paper A

with Svante Linusson

The numbers $\mathrm{av}_n^k(1324)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	1									
5	1	2	5	10	16	20	20	15	9	4	1					
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942

The numbers $\mathrm{av}_n^k(1324)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	1									
5	1	2	5	10	16	20	20	15	9	4	1					
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942

The numbers $\mathrm{av}_n^k(1324)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	1									
5	1	2	5	10	16	20	20	15	9	4	1					
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942

The numbers $\mathrm{av}_n^k(1324)$

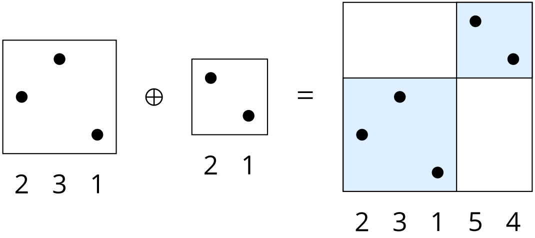
$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	1									
5	1	2	5	10	16	20	20	15	9	4	1					
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942

The numbers $\mathrm{av}_n^k(1324)$

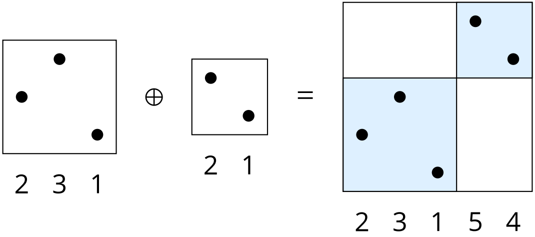
$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	1									
5	1	2	5	10	16	20	20	15	9	4	1					
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942

Paper A. What happens in the red region?

Permutations can be decomposed with respect to the *direct sum*:

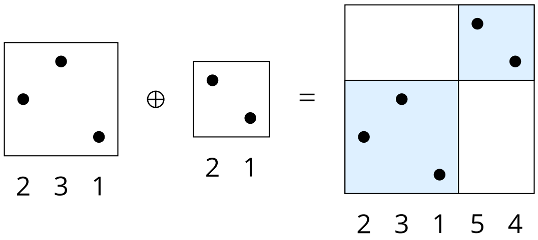


Permutations can be decomposed with respect to the *direct sum*:



Fact (CJS). Decomposable 1324-avoiders have a simple structure.

Permutations can be decomposed with respect to the *direct sum*:



Fact (CJS). Decomposable 1324-avoiders have a simple structure.

Fact (CJS). If $n \geqslant \mathrm{inv}(\pi) + 2$, then π is decomposable.

This is the key to the limit sequence.

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A)**.** $\pi \in S_n$ is *almost decomposable* if one of

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A)**.** $\pi \in S_n$ is *almost decomposable* if one of

$\pi,$

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A)**.** $\pi \in S_n$ is *almost decomposable* if one of

$$\pi, \ \pi \setminus 1,$$

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A)**.** $\pi \in S_n$ is *almost decomposable* if one of

$$\pi,\ \pi \setminus 1,\ \pi \setminus \pi(1),$$

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A). $\pi \in S_n$ is *almost decomposable* if one of

$$\pi,\ \pi \setminus 1,\ \pi \setminus \pi(1),\ \pi \setminus n,$$

Almost decomposable?

Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A)**.** $\pi \in S_n$ is *almost decomposable* if one of

$$\pi,\ \pi \setminus 1,\ \pi \setminus \pi(1),\ \pi \setminus n,\ \pi \setminus \pi(n)$$

is decomposable.

Almost decomposable?

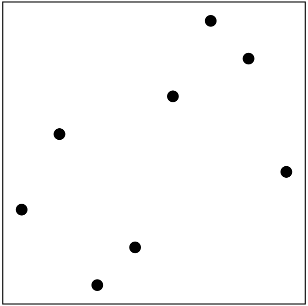
Question. Few inversions $\implies$ decomposable. What about slightly more inversions?

Definition (Paper A). $\pi \in S_n$ is *almost decomposable* if one of

$$\pi,\ \pi \setminus 1,\ \pi \setminus \pi(1),\ \pi \setminus n,\ \pi \setminus \pi(n)$$

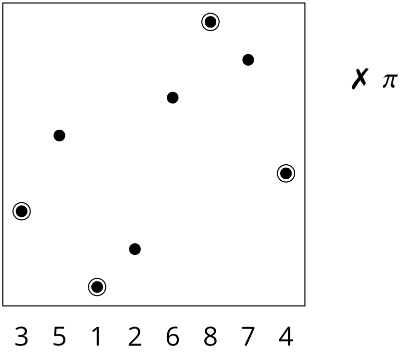
is decomposable.

Data suggests: every 1324-avoider in the red region is almost decomposable.

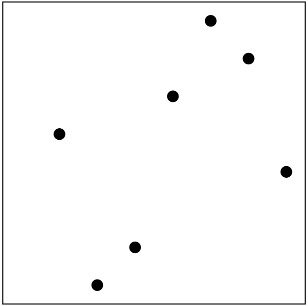


✗ π

3 5 1 2 6 8 7 4



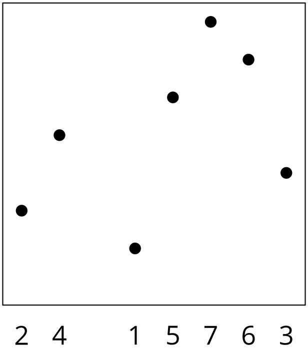
3 5 1 2 6 8 7 4
✗ π



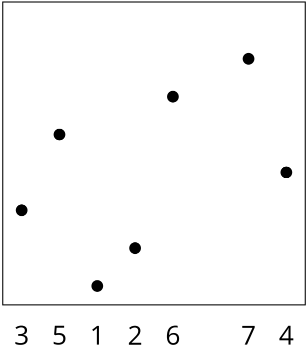
4 1 2 5 7 6 3

✗ π

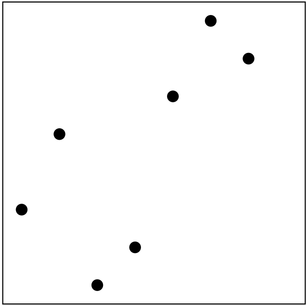
✗ $\pi \setminus \pi(1)$



✗ π
✗ $\pi \setminus \pi(1)$
✗ $\pi \setminus 1$



✗ π
✗ $\pi \setminus \pi(1)$
✗ $\pi \setminus 1$
✗ $\pi \setminus n$



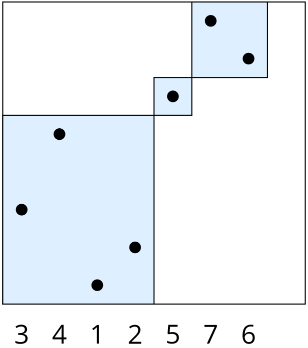
3 4 1 2 5 7 6

✗ π

✗ $\pi \setminus \pi(1)$

✗ $\pi \setminus 1$

✗ $\pi \setminus n$



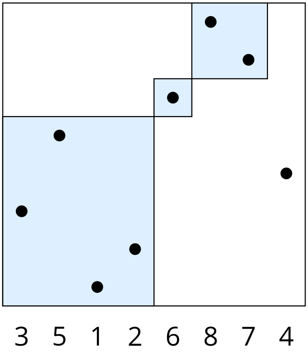
✗ π

✗ $\pi \setminus \pi(1)$

✗ $\pi \setminus 1$

✗ $\pi \setminus n$

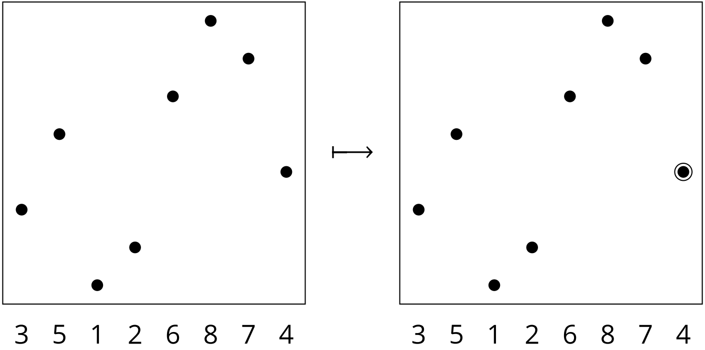
✓ $\pi \setminus \pi(n)$



✗ π
✗ $\pi \setminus \pi(1)$
✗ $\pi \setminus 1$
✗ $\pi \setminus n$
✓ $\pi \setminus \pi(n)$

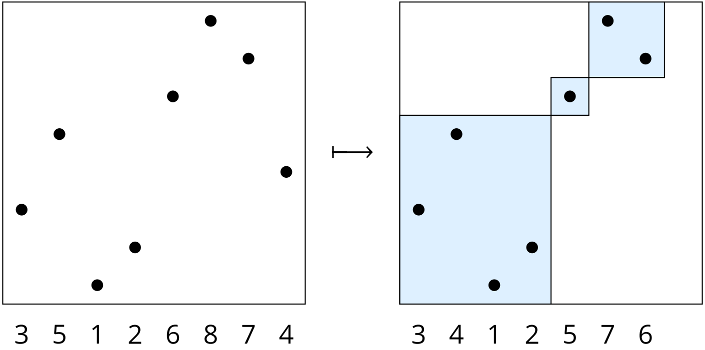
This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$



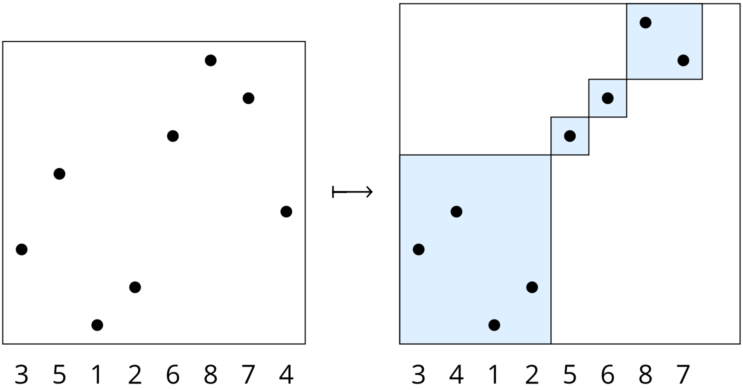
This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$



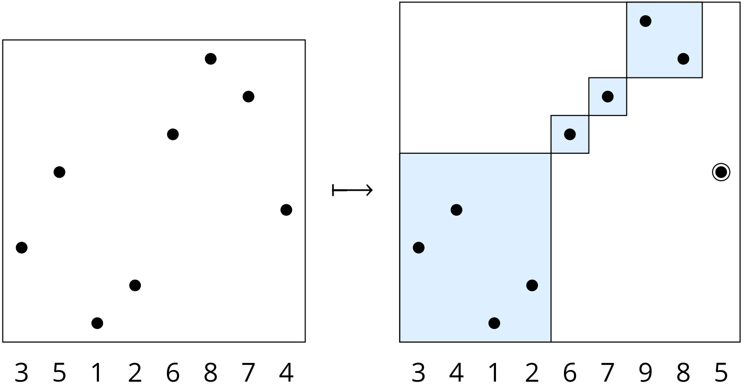
This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$



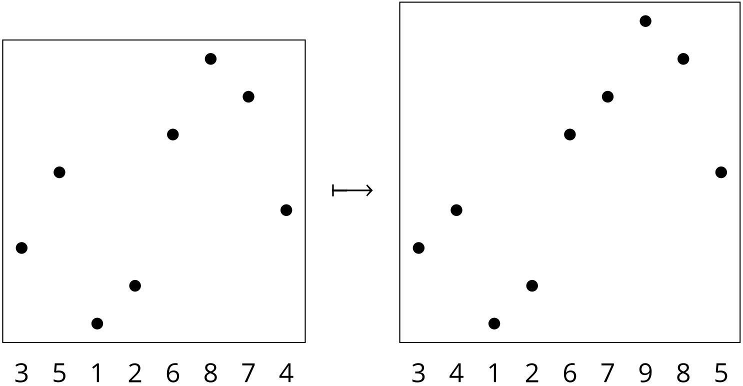
This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$



This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$



This structure allows us to define an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$

Theorem (Paper A)**.** If $n \geqslant \frac{k+7}{2}$, then every permutation in $\mathrm{Av}_n^k(1324)$ is almost decomposable.

Theorem (Paper A). If $n \geqslant \frac{k+7}{2}$, then every permutation in $\mathrm{Av}_n^k(1324)$ is almost decomposable.

Theorem (Paper A). If $n \geqslant \frac{k+7}{2}$, then

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k.

Theorem (Paper A). If $n \geqslant \frac{k+7}{2}$, then every permutation in $\mathrm{Av}_n^k(1324)$ is almost decomposable.

Theorem (Paper A). If $n \geqslant \frac{k+7}{2}$, then

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k.

Corollary (Paper A). 1324 is inversion monotone for $n \geqslant \frac{k+7}{2}$.

Paper B

with Anders Claesson, Svante Linusson,

and Henning Ulfarsson

Avoiding multiple patterns

The inversion monotonicity conjecture is too hard. Can we restrict the problem?

Avoiding multiple patterns

The inversion monotonicity conjecture is too hard. Can we restrict the problem?

For a set B of patterns,

$$\mathrm{Av}(B) = \{\text{permutations avoiding every pattern in } B\}.$$

All definitions extend naturally.

Avoiding multiple patterns

The inversion monotonicity conjecture is too hard. Can we restrict the problem?

For a set B of patterns,

$$\mathrm{Av}(B) = \{\text{permutations avoiding every pattern in } B\}.$$

All definitions extend naturally.

Question. When do we expect B to be inversion monotone?

Avoiding multiple patterns

The inversion monotonicity conjecture is too hard. Can we restrict the problem?

For a set B of patterns,

$$\mathrm{Av}(B) = \{\text{permutations avoiding every pattern in } B\}.$$

All definitions extend naturally.

Question. When do we expect B to be inversion monotone?

Question. Can we find inversion-monotone sets $B \ni 1324$?

1324 and a length-3 pattern

The patterns 123, 132 and 213 are contained in 1324, so $\mathrm{Av}(1324, 123) = \mathrm{Av}(123)$ and so on.

1324 and a length-3 pattern

The patterns 123, 132 and 213 are contained in 1324, so $\mathrm{Av}(1324, 123) = \mathrm{Av}(123)$ and so on.

The pair $\{1324, 321\}$ is **not** inversion monotone: we have

$$\mathrm{av}_{10}^{15}(1324, 321) = 60 > 52 = \mathrm{av}_{11}^{15}(1324, 321).$$

This is surprising!

1324 and a length-3 pattern

The patterns 123, 132 and 213 are contained in 1324, so $\mathrm{Av}(1324, 123) = \mathrm{Av}(123)$ and so on.

The pair $\{1324, 321\}$ is **not** inversion monotone: we have

$$\mathrm{av}_{10}^{15}(1324, 321) = 60 > 52 = \mathrm{av}_{11}^{15}(1324, 321).$$

This is surprising!

One interesting case: $\{1324, 231\}$ ($\{1324, 312\}$ is symmetrical).

1324 and a length-3 pattern

The patterns 123, 132 and 213 are contained in 1324, so $\mathrm{Av}(1324, 123) = \mathrm{Av}(123)$ and so on.

The pair $\{1324, 321\}$ is **not** inversion monotone: we have

$$\mathrm{av}_{10}^{15}(1324, 321) = 60 > 52 = \mathrm{av}_{11}^{15}(1324, 321).$$

This is surprising!

One interesting case: $\{1324, 231\}$ ($\{1324, 312\}$ is symmetrical).

> **Theorem** (Paper B)**.** The set $\{1324, 231\}$ is inversion monotone.

Proof. Very intricate injection.

Building inversion monotone sets

If B is a set of patterns, let

$$B' = \{\sigma : \sigma \setminus 1 \in B\},$$

i.e. 1 is inserted anywhere in π for every $\pi \in B$.

Building inversion monotone sets

If B is a set of patterns, let

$$B' = \{\sigma : \sigma \setminus 1 \in B\},$$

i.e. 1 is inserted anywhere in π for every $\pi \in B$.

Proposition (Paper B)**.** If B is inversion monotone, so is B'.

Sort of surprising, since B' grows very large if we iterate.

Building inversion monotone sets

If B is a set of patterns, let

$$B' = \{\sigma : \sigma \setminus 1 \in B\},$$

i.e. 1 is inserted anywhere in π for every $\pi \in B$.

Proposition (Paper B)**.** If B is inversion monotone, so is B'.

Sort of surprising, since B' grows very large if we iterate.

Corollary (Paper B)**.** Since 213 is inversion monotone, so is the set

$$\{1324, 3124, 3214, 3241\}.$$

Paper B: limit sequences of all pairs $\{1324, \sigma\}$, $\sigma \in S_4$.

σ	Inv-monotone	Limit sequence
1243	No	Partition numbers
2143	No	$2 \cdot$ partition numbers
1342	Conjectured	Overpartitions
1432	Conjectured	Partitions with a divider
2341	Conjectured	(g.f. of sand pile model)^2
2413	Conjectured	(g.f. of partitions)^2
2431	Conjectured	$P(x) \cdot$ (g.f. of 'steep' partitions)
3412	Conjectured	(g.f. of 'convex' penny arrangements[c])^2
3421	Conjectured	(g.f. of A115029)^2
4231	Conjectured	(g.f. of 'convex' partitions)^2
4321	Conjectured	(g.f. of partitions with 2 distinct parts)^2

Open problems.

- Extend almost-decomposability (see Note C).

Open problems.

- Extend almost-decomposability (see Note C).
- Prove that some pair $\{1324, \sigma\}$ with $\sigma \in S_4$ is inversion monotone.

Open problems.

- Extend almost-decomposability (see Note C).
- Prove that some pair $\{1324, \sigma\}$ with $\sigma \in S_4$ is inversion monotone.
- Understand *higher* limit sequences.

Open problems.

- Extend almost-decomposability (see Note C).
- Prove that some pair $\{1324, \sigma\}$ with $\sigma \in S_4$ is inversion monotone.
- Understand *higher* limit sequences.
- Prove half of the remaining half; iterate.

Thank you!

Open problems.

- Extend almost-decomposability (see Note C).
- Prove that some pair $\{1324, \sigma\}$ with $\sigma \in S_4$ is inversion monotone.
- Understand *higher* limit sequences.
- Prove half of the remaining half; iterate.