# Enumerating 1324-avoiders with few inversions

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Counting the 1324-avoiding permutations is notoriously difficult. We work towards this goal by enumerating $\mathrm{av}_n^k(1324)$, the number of 1324-avoiding $n$-permutations with exactly $k$ inversions, for all $k$ and $n \geqslant \frac{k+7}{2}$. The result follows from a new structural characterization of the permutations in question. Along the way, we prove half of a conjecture of Claesson, Jelínek and Steingrímsson (2012). Preprint: `arXiv:2408.15075`

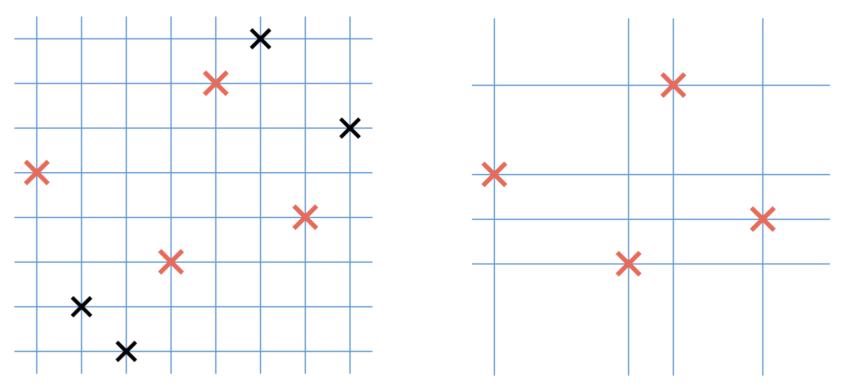
| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 1 | 2 | 2 | 1 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 1 | 2 | 5 | 6 | 5 | 3 | 1 | | | | | | | | | | | | | | | | | | | |
| 5 | 1 | 2 | 5 | 10 | 16 | 20 | 20 | 15 | 9 | 4 | 1 | | | | | | | | | | | | | | | |
| 6 | 1 | 2 | 5 | 10 | 20 | 32 | 51 | 67 | 79 | 80 | 68 | 49 | 29 | 14 | 5 | 1 | | | | | | | | | | |
| 7 | 1 | 2 | 5 | 10 | 20 | 36 | 61 | 96 | 148 | 208 | 268 | 321 | 351 | 347 | 308 | 241 | 165 | 98 | 49 | 20 | 6 | 1 | | | | |
| 8 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 106 | 171 | 262 | 397 | 568 | 784 | 1019 | 1264 | 1478 | 1628 | 1681 | 1619 | 1441 | 1173 | 866 | 574 | 338 | 174 | 76 |
| 9 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 181 | 286 | 443 | 664 | 985 | 1416 | 1988 | 2715 | 3589 | 4579 | 5631 | 6654 | 7559 | 8225 | 8545 | 8457 | 7930 | 7006 |
| 10 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 296 | 467 | 714 | 1077 | 1582 | 2305 | 3284 | 4617 | 6374 | 8665 | 11521 | 15012 | 19067 | 23599 | 28426 | 33300 | 37862 |
| 11 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 477 | 738 | 1127 | 1682 | 2477 | 3584 | 5134 | 7240 | 10100 | 13915 | 18976 | 25563 | 34017 | 44640 | 57739 | 73421 |
| 12 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 748 | 1151 | 1732 | 2577 | 3768 | 5450 | 7766 | 10976 | 15312 | 21171 | 28973 | 39338 | 52919 | 70657 | 93482 |
| 13 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1161 | 1756 | 2627 | 3868 | 5634 | 8098 | 11526 | 16216 | 22632 | 31266 | 42845 | 58213 | 78531 | 105137 |
| 14 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1766 | 2651 | 3918 | 5734 | 8282 | 11858 | 16786 | 23568 | 32768 | 45234 | 61902 | 84130 | 113477 |
| 15 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2661 | 3942 | 5784 | 8382 | 12042 | 17118 | 24138 | 33728 | 46776 | 64346 | 87939 | 119306 |
| 16 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2665 | 3952 | 5808 | 8432 | 12142 | 17302 | 24470 | 34298 | 47736 | 65916 | 90431 | 123184 |
| 17 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2665 | 3956 | 5818 | 8456 | 12192 | 17402 | 24654 | 34630 | 48306 | 66876 | 92001 | 125708 |
| 18 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2665 | 3956 | 5822 | 8466 | 12216 | 17452 | 24754 | 34814 | 48638 | 67446 | 92961 | 127278 |
| 19 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2665 | 3956 | 5822 | 8470 | 12226 | 17476 | 24804 | 34914 | 48822 | 67778 | 93531 | 128238 |
| 20 | 1 | 2 | 5 | 10 | 20 | 36 | 65 | 110 | 185 | 300 | 481 | 752 | 1165 | 1770 | 2665 | 3956 | 5822 | 8470 | 12230 | 17486 | 24828 | 34964 | 48922 | 67962 | 93863 | 128808 |

Table 1. The numbers $\mathrm{av}_n^k(1324)$.

## Preliminaries

A permutation $\pi \in S_n$ *contains* a pattern $p \in S_m$ if $\pi$ has a subsequence that is order-isomorphic to $p$. Otherwise $\pi$ *avoids* $p$. Let

$$\mathrm{Av}_n(p) = \{\pi \in S_n : \pi \text{ avoids } p\} \quad \text{and} \quad \mathrm{av}_n(p) = |\mathrm{Av}_n(p)|.$$



$\pi$ contains 3142

$\pi$ avoids 1324

We also set $\mathrm{Av}_n^k(p) = \{\pi \in \mathrm{Av}_n(p) : \mathrm{inv}(\pi) = k\}$ and $\mathrm{av}_n^k(p) = |\mathrm{Av}_n^k(p)|$.

## Background

The study of pattern avoiding permutations was initiated by Knuth (1968) with his characterization of the *stack-sortable* permutations as the 231-avoiders. In fact, for any length 3 pattern $p$,

$$\mathrm{av}_n(p) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the $n$-th Catalan number. However, for length 4 patterns the situation is much more complicated. There are three essentially distinct cases:

| Class | Enumeration | Asymptotics |
|---|---|---|
| $\mathrm{Av}(1234)$ | Complicated formula by Gessel (1990) | $9^n$ |
| $\mathrm{Av}(1342)$ | Complicated formula by Bóna (1997) | $8^n$ |
| $\mathrm{Av}(1324)$ | ? | $10.27^n < a_n < 13.5^n$ |

**Moral.** We know almost nothing about the 1324-avoiders.

One famous conjecture would provide an improvement.

**Conjecture 1** (Claesson–Jelínek–Steingrímsson 2012). For all nonnegative integers $n$ and $k$,

$$\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324).$$

**Remark.** If Conjecture 1 is true, then

$$\mathrm{av}_n(1324) < C^n,$$

where $C = e^{\pi\sqrt{2/3}} < 13.002$.

**Observation.** Conjecture 1 $\iff$ the columns of Table 1 are increasing.

## Main result

Many have tried to attack Conjecture 1, to no avail. It holds trivially in the light blue triangle of Table 1: the columns turn constant!

Our main result enumerates the values in the dark blue wedge of Table 1, thus proving Conjecture 1 for that region.

**Theorem 2** (Linusson–V. 2025). For all nonnegative integers $k$ and $n \geqslant \frac{k+7}{2}$,

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

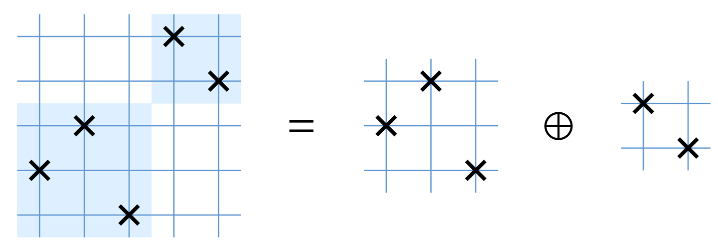
where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of $k$. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geqslant 0,$$

and this difference has a combinatorial interpretation.

## The ideas behind the proof

Permutations can be decomposed with respect to the *direct sum*:



$\Longrightarrow \quad \mathrm{comp}(23154) = 2.$

If $\pi \in \mathrm{Av}(1324)$ is decomposable, it must be of the form

$$\pi = \sigma \oplus 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \tau, \qquad \text{where } \sigma \in \mathrm{Av}(132),\ \tau \in \mathrm{Av}(213). \qquad (1)$$

An $n$-permutation with at most $n-2$ inversions is decomposable, so all permutations in the light triangle of Table 1 satisfy (1). Inversion sequences of 132-avoiders are partitions, so the sequence $1, 2, 5, 10, 20, 36, \ldots$ is $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$.

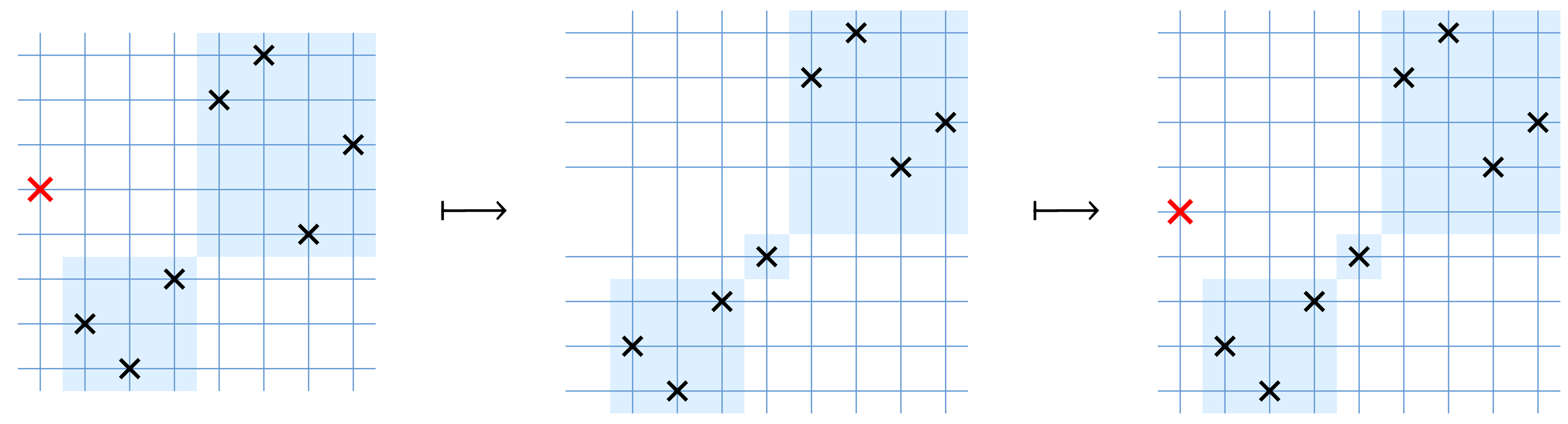
**Question.** 1324-avoiders with very few inversions have a very nice structure. What if we allow slightly more inversions?

**Definition 3** (Linusson–V. 2025). An $n$-permutation $\pi$ is called *almost decomposable* if it is indecomposable, but

$$\max\{\mathrm{comp}(\pi \setminus e) : e \in \{1, \pi_1, n, \pi_n\}\} \geqslant 2.$$

**Theorem 4** (Linusson–V. 2025). If $n \geqslant \frac{k+7}{2}$, then all permutations in $\mathrm{Av}_n^k(1324)$ are decomposable or almost decomposable.

Theorem 4 allows us to build an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$ when $n \geqslant \frac{k+7}{2}$.



We can extend this mapping naturally to *all* almost decomposable 1324-avoiders using symmetries of the square. The rest of the proof of Theorem 2 consists of a careful enumeration of the collection

$$\mathcal{R}_n^k := \mathrm{Av}_{n+1}^k(1324) \setminus f\big(\mathrm{Av}_n^k(1324)\big),$$

where $f$ is our injection. For example, any $\pi \in \mathrm{Av}_{n+1}^k(1324)$ with $\pi_1 = n+1$ is in $\mathcal{R}_n^k$.

**Question.** Can our method be extended to prove more of Conjecture 1, or to other problems in pattern avoidance?