Chow classes of matroids count standard Young tableaux

Jon Pål Hamre, Benjamin Schröter, Lorenzo Vecchi and **Emil Verkama** – KTH Royal Institute of Technology

We study the Chow classes of arbitrary matroids in the Grassmannian. Our main contribution identifies the Poincaré dual of the Chow class of a snake matroid with a specific ribbon Schur function. This extends to a standard Young tableaux count for the Chow class of any lattice path matroid, and by valuativity to a new expression for the Chow class for any matroid. Preprint: `arXiv:2511.01711`

What is a Chow class?

Let x be a point in the complex Grassmannian $G(k,n)$. $T = (\mathbb{C}^*)^n$ acts on $G(k,n)$ by scaling the coordinates. Consider the equivalence class

$$\left[\,\overline{Tx}\,\right] \in \underbrace{A^\bullet(G(k,n))}_{\text{Chow ring of } G(k,n)}.$$

Classical: $\overline{Tx} \cong$ projective toric variety of the base polytope of the matroid M_x.

$\left[\,\overline{Tx}\,\right]$ depends only on the matroid M_x of x. We call it the **Chow class** of M_x.

We can extend to a **valuative invariant** (= satisfies inclusion-exclusion on matroid base polytope subdivisions) defined on all matroids. Let

$$\mathcal{M}_{k,n} := \{\text{matroids of rank } k \text{ on } [n]\}.$$

Theorem 1 (Fink–Speyer 2012). There is a unique valuative matroid invariant

$$\mathrm{Sc} : \mathcal{M}_{k,n} \longrightarrow A^\bullet(G(k,n))$$

such that $\mathrm{Sc}(M_x) = \left[\,\overline{Tx}\,\right]$.

We have

$$A^\bullet(G(k,n)) \cong \mathrm{Sym} / \langle s_\eta \;:\; \eta \not\subseteq \underbrace{(n-k)^k}_{k\times(n-k) \text{ rectangle}} \rangle,$$

where s_η = Schur function. If M is connected, we can write

$$\mathrm{Sc}(M) = \sum_{\substack{\eta\subseteq(n-k)^k \\ \eta\vdash(k-1)(n-k-1)}} d_\eta(M)\, s_\eta .$$

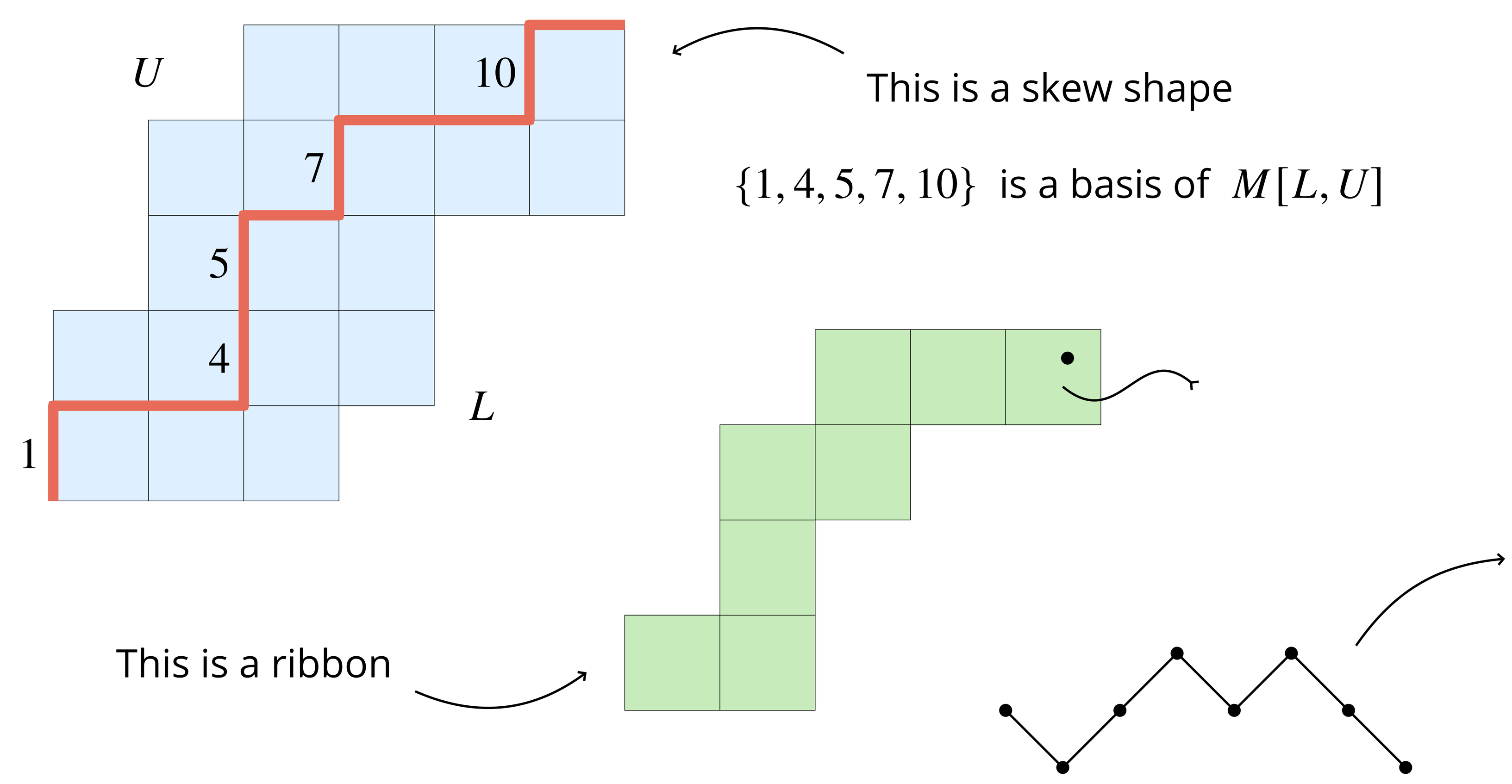
Conjecture 2 (Berget–Fink 2021). For any M and η, we have $d_\eta(M) \geqslant 0$.

This is true for representable matroids, but …

Issue. It is very difficult to compute $\mathrm{Sc}(M)$, even for representable M.

Lattice path matroids

Let L, U be two lattice paths from $(0,0)$ to $(n-k,k)$ with L never going above U. The **lattice path matroid** $M[L,U]$ has bases corresponding to lattice paths from $(0,0)$ to $(n-k,k)$ lying between L and U.



A **ribbon** is a connected skew shape that does not contain a 2×2 square.

The **snake matroid** S_ρ is the lattice path matroid of a ribbon shape ρ.

The **skew Schur function** for a skew shape λ/μ is

$$s_{\lambda/\mu} := \sum_{T\in \mathrm{SSYT}_{\lambda/\mu}} x^T = \sum_\eta c^\lambda_{\mu\eta}\, s_\eta .$$

Main results

Write η^c for the complement of η in $(n-k)^k$, and

$$\mathrm{Sc}^c(M) := \sum_\eta d_{\eta^c}(M)\, s_\eta .$$

Theorem 3. If ρ is a ribbon, then

$$\mathrm{Sc}^c(S_\rho) = s_\rho .$$

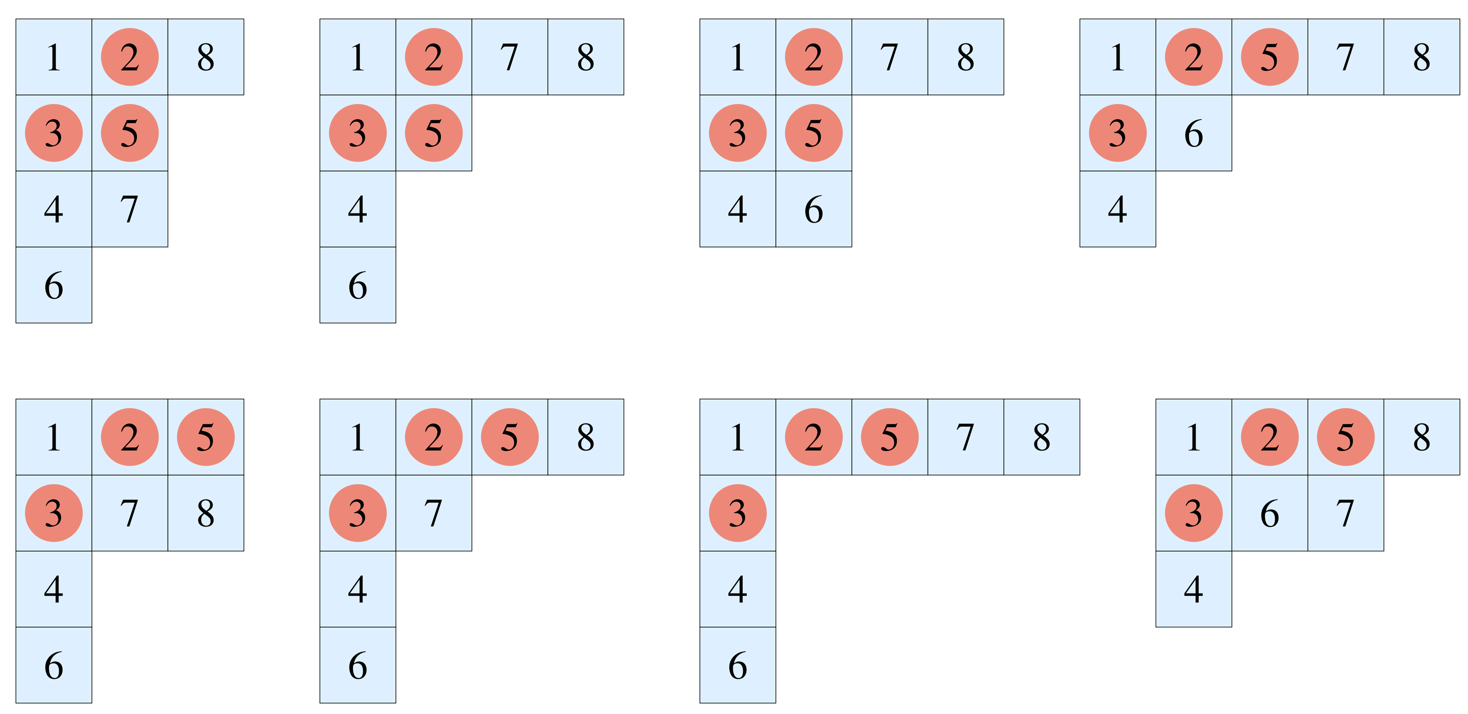
Gessel (1984) now tells us what the coefficients $d_\eta(S_\rho)$ are. For $b\in\mathbb{N}^k$, let

$$\rho(b) := \text{ribbon with } i\text{-th row from the bottom of length } b_i,$$

$$\mathrm{SYT}_\eta(b) := \left\{ T\in \mathrm{SYT}_\eta \;:\; \mathrm{Des}(T) = \left\{ \sum_{i=1}^m b_i \;:\; m\in[k-1] \right\} \right\}. \qquad (*)$$

Corollary 4. If $\rho = \rho(b)$ is a ribbon, then $d_{\eta^c}(S_\rho) = |\mathrm{SYT}_\eta(b)|$.

Example. Let $b = (2,1,2,3)$. The SYT with descent set $\{2,3,5\}$ are as follows.



$$\mathrm{Sc}^c\left(S_{\rho(b)}\right) = s_{(3,2,2,1)} + 2\, s_{(4,2,1,1)} + s_{(4,2,2)} + s_{(5,2,1)} + s_{(3,3,1,1)} + s_{(5,1,1,1)} + s_{(4,3,1)} .$$

For a connected lattice path matroid M of shape λ/μ,

$$\mathrm{Sc}(M) = \sum_{\rho\subseteq\lambda/\mu} \mathrm{Sc}(S_\rho) .$$

Let S_U and S_L be the snake matroids of the upper and lower boundaries of λ/μ, with associated descent sets $\{c_1 < \ldots < c_{k-1}\}$ resp. $\{d_1 < \ldots < d_{k-1}\}$. Let

$$\mathcal{A}_{\lambda/\mu} := \{[c_i, d_i] : i\in[k-1]\} .$$

Theorem 5. For a connected lattice path matroid M of shape λ/μ, we have

$$d_{\eta^c}(M) = \left|\left\{ T\in\mathrm{SYT}_\eta \;:\; \mathrm{Des}(T) \text{ is a transversal of } \mathcal{A}_{\lambda/\mu} \right\}\right| .$$

Also. We prove Conjecture 2 for certain M and η, and we can compute $\mathrm{Sc}(M)$ for much larger M.

RSK and volume

Knauer–Martínez-Sandoval–Ramírez Alfonsín (2018):

$$\mathrm{V}\left(S_{\rho(b)}\right) = |\{\pi\in S_{n-1} \;:\; \mathrm{Des}(\pi) \text{ is as in } (*)\}| .$$

Hamre (2025): for any matroid M, we have

$$\mathrm{V}(M) = \sum_\eta |\mathrm{SYT}_\eta|\, d_{\eta^c}(M) .$$

So, we recover Theorem 6 from Corollary 4 using RSK! We can generalize.

Theorem 6. For a connected lattice path matroid M of shape λ/μ, we have

$$\mathrm{V}(M) = \left|\left\{ \pi\in S_{n-1} \;:\; \mathrm{Des}(\pi) \text{ is a transversal of } \mathcal{A}_{\lambda/\mu} \right\}\right| .$$